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OPTIMIZATION OF THE OBSERVATION PROCESS*

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The special problem of designing the observation process in cases when the observer parameters depend on the trajectory of the controlled dynamic system is considered. Such a dependence arises, for instance, when the measuring device is installed on a controlled moving platform (an aircraft) or if its parameters are affected by the dynamically varying characteristics of the environment (temperature). It is interesting to examine the selection of the trajectory of a dynamic system that minimizes the maximum possible estimation error (the size of the information set) /1, 2/. In formal terms, this question can be reduced to an optimal control problem with a non-smooth functional of a special form. Necessary conditions of optimality are given and some optimal observation processes are constructed.

Although the problem considered in this paper may be regarded as an infinite-dimensional generalization of some regression experiment design problem /3/, the results appear to be new and in a certain sense unexpected. Control of the size of the information set was previously considered in /4-6/.

1. Statement of the problem. The observed signal is given by

$$y(t) = a^*(t)\theta + \xi(t),\ t \in [t_0, T] \tag{1.1}$$

where $\theta \in R^n$ is an unknown parameter vector, and $a(t) \in R^n$ is a known vector function whose components $a^i(\cdot)$ are assumed to be linearly independent and continuous on $[t_0, T]$; the unknown scalar disturbances $\xi(t)$ are bounded,

$$\langle \xi^2 \rangle \leqslant 1 \ \left(\langle f \rangle = \int_{t_0}^{T} f(t)\,dt\right) \tag{1.2}$$

Here and henceforth, the asterisk denotes the transpose and $i = 1, 2, \ldots, n$.

For a fixed $y(\cdot)$ the set of vectors θ that satisfy (1.1), (1.2) is called an information set compatible with the realized signal /2/. In our case, the information set is an ellipsoid

$$E(\theta^\circ, P) = \{\theta \in R^n : (\theta - \theta^\circ)^* P (\theta - \theta^\circ) \leqslant 1 - h^2\} \tag{1.3}$$
$$\theta^\circ = P^{-1}d,\ P = P(a(\cdot)) = \langle aa^* \rangle$$
$$d = \langle ay \rangle,\ h^2 = \langle y^2 \rangle - d^* P^{-1} d$$

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The size of the information ellipsoid $E(\theta^\circ, P)$ is determined by the matrix P and the quantity h^2, which depends on the realized signal. If the estimator is the centre of the ellipsoid θ°, then the maximum possible estimation errors is identical with the maximum eigenvalue of the matrix P^{-1}.

Remark. The vector θ° is identical with the ordinary least squares estimator for the signal (1.1). The spectrum of the matrix P determines the error variance if the disturbances are described by stationary stochastic processes /7/.

In what follows we assume that $a(\cdot)$ is the solution of the linear differential equation

$$a^\cdot(t) = A(t)\,a(t) + B(t)\,u(t),\ t \in [t_0,\ T],\ a(t_0) = a_0, \tag{1.4}$$

where $A(t)$ and $B(t)$ are continuous $(n \times n)$ and $(n \times m)$ matrices in $[t_0,\ T]$ and system (1.4) is completely controllable. As the admissible controls $u(\cdot)$ we choose measurable function that satisfy the constraint

$$u(\cdot) \in U \subset L_2^m[t_0,\ T] \tag{1.5}$$

Here U is a convex weakly compact set, identified in particular with an ellipsoid in the space $L_2^m[t_0, T]$,

$$U = \{u(\cdot) : \langle u^* R u\rangle \leqslant \Delta^2\} \tag{1.6}$$

where $R(t)$ is a symmetrical positive definite matrix in $[t_0,\ T]$.

We can then consider the following problem.

Problem 1. Find an admissible control $u_0(\cdot)$ that satisfies the constraint (1.5) and the corresponding solution $a_0(\cdot)$ of Eq.(1.4) that minimizes the maximum eigenvalue of the matrix P^{-1}.

For convenience, we consider in what follows the equivalent problem of maximizing the minimum eigenvalue of the matrix P.

2. ***Necessary conditions of optimality.*** *Theorem 2.1.* Let $u_0(\cdot)$, $a_0(\cdot)$ be a solution of Problem 1, $P_0 = P(a_0(\cdot))$. Then there is a symmetrical non-negative definite matrix M such that

$$\langle p^* B u_0\rangle = \max_{u(\cdot)\in U} \langle p^* B u\rangle \tag{2.1}$$

$$p^\cdot(t) = -A^*(t)\,p(t) - M a_0(t),\ t \in [t_0,\ T],\ p(T) = 0 \tag{2.2}$$

The matrix M can be represented in the form

$$M = \sum_{k=1}^{l} \nu_k \psi_k \psi_k^*,\quad \nu_k \geqslant 0,\quad \sum_{k=1}^{l} \nu_k = 1,\quad l \leqslant \frac{n(n+1)}{2} + 1 \tag{2.3}$$
$$\psi_k \in \Psi_0 = \{\psi \in R^n : P_0\psi = \lambda_0\psi\},\ k = 1,\ 2,\ \ldots,\ l$$

where $\lambda_0 = \lambda_{\min}(P_0)$ is the minimum eigenvalue of the matrix P_0.

The proof of the theorem is based on standard methods of external problems and uses the fact that the functional $\varphi(a(\cdot)) = \lambda_{\min}(P(a(\cdot)))$ is quasidifferentiable in the sense of /8/.

Corollary 2.1. Let $u_0(\cdot), a_0(\cdot)$ be a solution of Problem 1 and let the set of admissible controls U be defined by (1.6). Then

$$\rho u_0(t) = R^{-1}(t)\,B^*(t)\,p(t),\ t \in [t_0,\ T],\ \rho > 0 \tag{2.4}$$

where $p(t)$ is defined by (2.2), (2.3).

Let I be the identity matrix, $m = n$, and

$$A(t) = 0,\ R(t) = I,\ B(t) = I,\ t \in [t_0,\ T],\ t_0 = 0 \tag{2.5}$$

In what follows, we will always assume that conditions (1.6) and (2.5) are satisfied with respect to Problem 1. In this case, Problem 1 has certain symmetry properties that make it possible to obtain the solution in analytical form.

Invariance of the set U and the spectrum of the matrix P relative to orthogonal transformations leads to the following proposition.

Lemma 2.1. Let $a_0(\cdot)$ be a solution of Problem 1, and S an arbitrary orthogonal matrix. Then $Sa_0(t)$ is a solution of the same problem with the vector Sa_0 substituted for a_0 in (1.4).

Corollary 2.2. Let $a_0(\cdot)$ be a solution of Problem 1 for $a_0 = 0$. Then $Sa_0(\cdot)$ is also a solution of Problem 1 for any orthogonal matrix S.

Lemma 2.2. Let $a_0(\cdot)$ be a solution of Problem 1, and Ψ_0 the set of eigenvectors of the matrix $P_0 = P(a_0(\cdot))$ corresponding to the minimum eigenvalue. Then

$$\dim\{\Psi_0 \cup \{a_0\}\} = n$$

Proof. Assume that the lemma is not true. Then there exists a vector $\eta \neq 0$ such that $\eta^* a_0 = 0$, $\eta^*\psi = 0$, $\psi \in \Psi_0$, and therefore by (2.2)-(2.4) $\eta^* a_0(t) = 0$, $t \in [0, T]$. The components $a_0^i(\cdot)$ are thus linearly dependent on $[0, T]$ and the matrix P_0 is singular. We can show that this contradicts the optimality of $a_0(\cdot)$.

Thus, the eigenvalues of the optimal matrix P_0, with the possible exception of the maximum eigenvalue, are all equal. Geometrically this means that $E(\theta^\circ, P_0)$ is an ellipsoid of revolution .

Corollary 2.3. If $a_0 = 0$, then $P_0 = \lambda_0 I$, $\lambda_0 > 0$.

3. *Constructing the solution.* *Theorem 3.1*. The solution $a_0(\cdot)$ of Problem 1 can be represented in the form

$$a_0(t) = Sa_*(t), \quad a_*^i(t) = c_i \sin(\omega_i t + \varphi_i) \tag{3.1}$$

$$\omega_i T + \varphi_i = {}^1/_2 (2k_i - 1)\pi$$

where k_i are natural numbers, and S is an arbitrary orthogonal matrix that satisfies the condition $Sa_*(0) = a_0$.

Proof. From (2.2) and (2.4) it follows that in this case

$$a_0^{\cdot\cdot}(t) + Ma_0(t) = 0, \quad a^{\cdot}(T) = 0 \tag{3.2}$$

Since M is a symmetrical non-negative definite matrix, an orthogonal matrix Q exists such that

$$Q^* M Q = \operatorname{diag}\{\omega_1^2, \omega_2^2, \ldots, \omega_n^2\}$$

Let $a_*(t) = Q^* a_0(t)$. Then from (3.2) we obtain an equation and a boundary condition for $a_*^i(t)$. The proof is completed by reference to Lemma 2.1.

Theorem 3.2. Let $a_0(\cdot)$ be a solution of Problem 1 for $a_0 = 0$. Then an orthogonal matrix S exists such that (3.1) hold for $\varphi_i = 0$ and

$$\omega_i = \frac{(2i-1)\pi}{2T}, \quad c_i = c, \quad c^2 = \frac{24\Delta^2 T}{\pi n (4n^2 - 1)} \tag{3.3}$$

Proof. Since $a_0 = 0$, we obtain by Theorem 3.1 that the solution of Problem 1 can be represented in the form (3.1) and $\varphi_i = 0$. By (1.6) and Corollary 2.3, we obtain

$$\langle (a_*^i)^2\rangle = {}^1/_2 c_i^2 T = \lambda_0 \tag{3.4}$$

$$\langle a_*^i a_*^j\rangle = 0; \; i \neq j; \; \sum \langle (a_*^{\cdot i})^2\rangle \leqslant \Delta^2$$

Here and henceforth, summation is from $i = 1$ to $i = n$.

Hence

$$c_i = c_j, \; \omega_i \neq \omega_j, \; i \neq j; \; \sum {}^1/_2 c_i^2 T \omega_i^2 = \lambda_0 \sum \omega_i^2 \leqslant \Delta^2$$

and therefore the maximum eigenvalue λ_0 is attained for ω_i that satisfy the first relationship in (3.3). In this case, $c_i^2 = 2\lambda_0 / T = 2\Delta^2 / (T \sum \omega_i^2)$ and we obtain the last two relationships in (3.3). The theorem is proved.

For $a_0 \neq 0$, the determination of c_i, ω_i, and φ_i in (3.1) is somewhat more difficult. Let

$$z > 0, \quad f_i(z) = \left(\frac{1}{\cos^2 x_i} + \frac{z}{x_i^2}\right)^{-1}, \quad g_i(z) = \frac{x_i^2}{\cos^2 x_i} - z$$

where $x_i = x_i(z)$ is the i-th (in increasing order) non-negative solution of the equation

$$x \operatorname{tg} x = z \tag{3.5}$$

The following lemma is given without proof.

Lemma 3.1. The equation

$$\sum f_i(z)(\alpha - g_i(z)) = 0 \tag{3.6}$$

has a unique non-negative solution for any $\alpha \geqslant \pi^2 n(n-1)/3$.

Theorem 3.3. Let $0 < \| a_0 \|^2 \leqslant \varkappa_n$, $\varkappa_n = 6\Delta^2 T / [\pi^2 n (n-1)]$ and let z_0 be the solution of Eq.(3.6) for $\alpha = 2\Delta^2 T / \| a_0 \|^2$. Then the solution $a_0(\cdot)$ of Problem 1 can be represented in the form

$$a_0(t) = Sa_*(t), \quad a_*^i(t) = c_i \cos(x_i^\circ (t/T - 1)) \tag{3.7}$$

$$c_i^2 = 4\frac{(\Delta^2 T + z_0 \| a_0 \|^2) x_i^\circ}{(2x_i^\circ + \sin 2x_i^\circ) \Sigma (x_i^\circ)^2}, \quad x_i^\circ = x_i(z_0)$$

Let us outline the proof of the theorem, omitting some elementary but lengthy transformations. First note that the matrices M and P_0 commute. Therefore the transformation Q in the proof of Theorem 3.1 can be chosen so that it simultaneously diagonalizes P_0. By Lemma 2.2, all the eigenvalues of the matrix P_0, with the possible exception of the maximum eigenvalue, are equal.

Therefore, for $a_*^i(\cdot)$ that satisfy the first two equalities in (3.1) we have the last two relationships in (3.4) and

$$\langle (a_*^1)^2 \rangle = \lambda_1 \geqslant \lambda_0 = \langle (a_*^m)^2 \rangle, \quad m = 2, 3, \ldots, n \tag{3.8}$$

Moreover, by (2.1), the last inequality in (3.4) may be replaced with an equality. Evaluating the corresponding integrals, we can show that the second relationship in (3.4) leads to the condition

$$\omega_i \neq \omega_j, \quad i \neq j, \quad \omega_i T \operatorname{tg} \omega_i T = z$$

and $\omega_i T$ are identical with the first (in increasing order) non-negative solutions of Eq.(3.5).

Under the conditions of the theorem, (3.8) also reduces to an equality, and the determination of λ_0, z, c_i involves solving the system of transcendental equations

$$c_i^2 T \cos^2 x_i(z) = 2f_i(z)\lambda_0$$

$$\Sigma c_i^2 \cos^2 x_i(z) g_i(z) = 2T\Delta^2, \quad \Sigma c_i^2 \cos^2 x_i(z) = \| a_0 \|^2$$

After some transformations, the solution of this system can be represented in the form (3.7).

It is interesting to obtain an expression for the optimal solution when $\| a_0 \|^2 = \varkappa_n$. In this case, $z_0 = 0$, $x_i^\circ = (i-1)\pi$, and from (3.7) we obtain

$$a_*^1(t) = c_1, \quad a_*^m(t) = c_m \cos((m-1)\pi(t/T - 1)) \tag{3.9}$$

$$2c_1^2 = c_m^2 = 2\varkappa_n / (2n-1), \quad m = 2, 3, \ldots, n$$

In this case, addition of a constant to $a_*^1(t)$ obviously does not violate the last two relationships in (3.4) and does not alter the minimum eigenvalue λ_0 of the matrix P_0. The form of the optimal solution thus does not change for $\| a_0 \|^2 > \varkappa_n$.

Theorem 3.4. For $\| a_0 \|^2 \geqslant \varkappa_n$ the solution of Problem 1 can be represented in the form $a_0(t) = Sa_*(t)$, where S is an orthogonal matrix and $a_*(t)$ satisfies relationships (3.9) for

$$c_1^2 = \| a_0 \|^2 - c_2^2 - c_3^2 - \ldots - c_n^2 = \| a_0 \|^2 - 2\varkappa_n (n-1) / (2n-1)$$

Remark. As $\alpha \to +\infty$, the solution z_0 of Eq.(3.6) tends to $\infty+$ and $x_i(z_0) \to {}^1/_2 (2i-1)\pi$. Theorems 3.2-3.4 can therefore be reduced to a single proposition.

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